

INJECTIVE MODULES OVER SOME RINGS OF DIFFERENTIAL OPERATORS

TONY J. PUTHENPURAKAL

ABSTRACT. Let R be a regular domain containing a field K of characteristic zero and let D be the ring of K -linear differential operators on R . Let E be an injective left D -module. We ask the question, when is E injective as a R -module? We show that this is indeed the case when $R = K[X_1, \dots, X_n]$ or $R = K[[X_1, \dots, X_n]]$ or $R = \mathbb{C}\{z_1, \dots, z_n\}$. We also give an application of our result to local cohomology.

INTRODUCTION

The motivation for this paper comes from a problem in local cohomology which we now describe. Let K be a field of characteristic zero, $R = K[X_1, \dots, X_n]$ and let I be an ideal in R . Let $A_n(K) = K \langle X_1, \dots, X_n, \partial_1, \dots, \partial_n \rangle$ be the n^{th} Weyl algebra over K . By a result due to Lyubeznik, see [6], the local cohomology modules $H_I^i(R)$ are finitely generated $A_n(K)$ -modules for each $i \geq 0$. If J is another ideal in R then we have a Mayer-Vietoris sequence

$$\cdots \rightarrow H_{I+J}^i(R) \xrightarrow{\rho^i} H_I^i(R) \oplus H_J^i(R) \xrightarrow{\pi^i} H_{I \cap J}^i(R) \xrightarrow{\delta^i} H_{I+J}^{i+1}(R) \rightarrow \cdots$$

It can be easily seen that for all $i \geq 0$; ρ^i, π^i are in-fact $A_n(K)$ -linear; for instance see [11, 1.5]. A natural question is whether δ^i is $A_n(K)$ -linear for all $i \geq 0$? In this paper we show this is so; see Proposition 3.3. The crucial ingredient is the following: Let E be an injective left $A_n(K)$ module and since R is a subring of $A_n(K)$ consider E as a R -module. Then we prove that E is an injective R -module. More generally we prove the following result:

Theorem 1. *Let S be a ring containing a regular commutative ring R as a subring. Assume that S considered as a right R -module is projective. Let E be a left S -module which is injective as a S -module. Then E is an injective R -module.*

Our motivation of S in the theorem above is ring's of differential operators. Note that when $R = K[X_1, \dots, X_n]$ or $R = K[[X_1, \dots, X_n]]$ or $R = \mathbb{C}\{z_1, \dots, z_n\}$ and S is the ring of differential operators on R then S satisfies the hypotheses of Theorem 1, see section 1. After we proved the result, we observed that there are many other examples of rings which satisfy the hypotheses of Theorem 1. In section 1 we have listed many examples of S which satisfy the hypotheses in Theorem 1.

The main technical tool used to prove Theorem 1 is the following result:

Date: December 11, 2013.

1991 Mathematics Subject Classification. Primary 13N10; Secondary 13C11, 13D45.

Key words and phrases. D-modules, local cohomology, associated primes, injective modules.

Lemma 2. *Let R be a regular commutative ring and let E be a R -module. If $\text{Ext}_R^1(R/J, E) = 0$ for every ideal J generated by a regular sequence, then E is an injective R -module.*

Matlis theory of injective modules over left Noetherian rings, in particular over commutative Noetherian rings is a very basic tool. We analyze the structure of injective S -modules considered as a R -module. For this we first assume that S is a left Noetherian ring. We make a further assumption:

(*) Given an ideal I in R and $s \in S$, there exists $r \geq 1$ (r depending on s) such that $I^r s \subseteq SI$.

This hypothesis is satisfied when S is the ring of differential operators. It is also trivially satisfied when S is commutative. We show

Theorem 3. *(with assumptions as above). Let M be an S -module with $E_S(M)$ an indecomposable injective S -module. Then $\text{Ass}_R E_S(M)$ has a unique maximal element P . Furthermore $P \in \text{Ass}_R M$.*

A particularly interesting case is when S is commutative and finitely generated as a R -module. Note that we are also assuming that S is projective as a R -module. This case is equivalent to assuming S is Cohen-Macaulay as a ring. We prove

Theorem 4. *(with assumptions as above). Let \mathfrak{m} be a maximal ideal of S . Set $\mathfrak{n} = \mathfrak{m} \cap R$. Let $\mathfrak{n}S = Q_1 \cap Q_2 \cap \cdots \cap Q_c$ be a minimal primary decomposition of $\mathfrak{n}S$ where Q_1 is \mathfrak{m} -primary. Then*

$$E_S\left(\frac{S}{\mathfrak{m}}\right) = E_R\left(\frac{R}{\mathfrak{n}}\right)^{\ell_R(S/Q_1)}.$$

We now describe in brief the contents of the paper. In section 1 we give many examples of rings S which satisfy the hypotheses for Theorem 1. In section 2 we prove Lemma 2. In section 3 we prove Theorem 1. In section 3 we also give application of our result in the theory of local cohomology; as described in the beginning of this section. In section 4 we prove Theorem 3. In section 5 we prove Theorem 4.

1. EXAMPLES

In this section we give many examples of rings S which satisfy the hypotheses for Theorem 1.

Example 1: Let K be a field of characteristic zero, $R = K[X_1, \dots, X_n]$ and let $S = A_n(K) = K \langle X_1, \dots, X_n, \partial_1, \dots, \partial_n \rangle$ be the n^{th} Weyl algebra over K . By [2, p. 3, 1.2] it follows that every $s \in S$ has a unique expression

$$s = \sum_{\alpha \in \mathbb{N}^n} \phi_\alpha(X) \partial^\alpha \quad \text{with } \phi_\alpha(X) \in R.$$

In other words S is free as a left R -module. The fact that S is also free as a right R -module is also known. However due to lack of a reference we give a proof here.

(a) Using the defining relations of the Weyl algebra it is clear that any $s \in S$ can be written as

$$(\dagger) \quad s = \sum_{\alpha \in \mathbb{N}^n} \partial^\alpha \psi_\alpha(X) \quad \text{with } \psi_\alpha(X) \in R.$$

(b) We prove that the above expression is unique. For this we need the following Lemma which is easy to prove

Lemma 1.1. *Let A be a ring containing a field K and let g be a K -linear derivation on R . Then in the ring $D_K(A)$ of K -linear differential operators, for any a in A we have*

$$g^m a = \sum_{i=0}^m \binom{m}{i} g^{m-i}(a) g^i. \quad \square$$

So let $s = \sum_{\alpha \in \mathbb{N}^n} \partial^\alpha \psi_\alpha(X) = 0$. If possible assume that some $\psi_\alpha(X) \neq 0$. Let $\alpha^* \in \mathbb{N}^n$ be such that $\psi_{\alpha^*}(X) \neq 0$ and $\psi_\alpha(X) = 0$ for all α with $|\alpha| > |\alpha^*|$. Clearly $\alpha^* \neq 0$. Also using 1.1 we get that

$$s = \sum_{\alpha^*} \psi_{\alpha^*}(X) \partial^{\alpha^*} + \sum_{|\alpha| < |\alpha^*|} c_\alpha \partial^\alpha.$$

Since $s = 0$ it follows that $\psi_{\alpha^*}(X) = 0$ for all α^* , a contradiction. Thus the expression in (†) is unique and so S is free as a right R -module.

Example 2: Let K be a field of characteristic zero, $R = K[[X_1, \dots, X_n]]$ and let $S =$ ring of K -linear differential operators on R . Then $S = R[\partial_1, \dots, \partial_n]$. Every $s \in S$ has a unique expression

$$s = \sum_{\alpha \in \mathbb{N}^n} \phi_\alpha(X) \partial^\alpha \quad \text{with } \phi_\alpha(X) \in R.$$

In other words S is free as a left R -module. By an argument similar to that in Example 1 we can show that there every $s \in S$ has a unique expression

$$s = \sum_{\alpha \in \mathbb{N}^n} \partial^\alpha \psi_\alpha(X) \quad \text{with } \psi_\alpha(X) \in R.$$

So S is free as a right R -module.

Example 3: $R = \mathbb{C}\{z_1, \dots, z_n\}$ the ring of convergent power series with complex coefficients. Let $S =$ ring of \mathbb{C} -linear differential operators on R . Then $S = R[\partial_1, \dots, \partial_n]$. As in Example 2 we can prove that S is free as a right R -module.

Example 4: R any commutative regular ring and $S = R[X_1, \dots, X_n]$. Then clearly S is free as a right R -module.

Example 5: R any commutative regular ring and $S = M_n(R)$; the ring of $n \times n$ matrices over R . Clearly $S \cong R^{n^2}$ as a right R -module. The ring R can be considered as a subring of R via scalar matrices. Note that $R \subseteq Z(S)$; the center of S .

Example 6: (A non-Noetherian example) Let $R = \mathbb{Z}$ and let $S \subseteq \mathbb{C}$ be the subring of all algebraic integers. Then S is torsion-free R -module, so it is free.

Example 7: (Differential polynomial rings) R any commutative regular ring and let $\delta: R \rightarrow R$ be a derivation on R . Consider the differential polynomial ring $S = R[X; \delta]$. An element s in S has a unique expression $s = \sum_{i=0}^m a_i X^i$. So S is a free left R -module. Note that $Xa = aX + \delta a$. We can prove that

$$X^n a = \sum_{i=0}^n \binom{n}{i} \delta^i(a) X^{n-i}.$$

So by an argument similar to in Example 1 we can show that each element s in S has a unique expression $s = \sum_{i=0}^m X^i b_i$. It follows that S is free as a right R -module.

Example 8: (Group rings) R any commutative regular ring and let G be a group. Consider the group ring $S = R[G] = \bigoplus_{\sigma \in G} R\sigma$. Notice $r\sigma = \sigma r$ for any $r \in R$

and $\sigma \in G$. It follows that $R[G] = \bigoplus_{\sigma \in G} \sigma R$. It follows that S is free as a right R -module. Also note that $R \subseteq Z(S)$.

Example 9: Let K be a field of characteristic zero, $A = K[X_1, \dots, X_n]$ and let $f \in A$ be a non-constant polynomial. Set $R = A_f$ and let S be the ring of K -linear differential operators on R . Then $S = R[\partial_1, \dots, \partial_n]$. As in Example 1 we can prove that S is a free as a right R -module.

Example 10: Let S be a Cohen-Macaulay affine algebra over an infinite field K and let R be a Noether normalization of S . Then R is a regular subring of S and since S is Cohen-Macaulay, S is projective as a R -module. As all projectives over R are free we get that S is free as a R -module.

In Examples 1-10, S was a free right R -module. We now give two examples when S is only projective as a R -module.

Example 11: Let R be a Dedekind domain with quotient field K and let L be a finite field extension of K . Set S to be the integral closure of R in L . Then S also a Dedekind ring. In general S is a projective R -module and not-necessarily free. In fact if K is a number-field with R not a P.I.D then there always exists a finite extension L with S not-free as a R -module; see [7]. For a specific example see [10]. The author thanks Sudhir Ghorpade for this specific example.

Example 12: Let R be a regular domain having a projective module N such that $S = \text{Hom}_R(N, N)$ is not free as a R -module. Clearly S is projective as a R -module. Also note that R can be considered as a subring of S via the map $i: R \rightarrow S$ where $i(r)$ is the multiplication map. Clearly R is in the center of S . The following specific example was constructed by Manoj Keshari. Recall a projective module P is said to be cancellative if $P \oplus R^n \cong Q \oplus R^n$ implies $P \cong Q$. Let A be the homogeneous localization $\mathbb{R}[X, Y, Z]_{(X^2+Y^2+Z^2)}$ and let $R = A_0$. Then R is a smooth affine surface. The projective module $K_R \oplus R$ is not cancellative (here K_R denotes the canonical module of R); see [1, Example 3.1]. Since $K_R \oplus R$ is not cancellative then there exists a projective module P with $K_R \oplus R \oplus R^r \cong P \oplus R^r$, but $P \not\cong K_R \oplus R$. It can be shown that $\text{Hom}_R(P, P)$ is not free as a R -module.

2. PROOF OF LEMMA 2

In this section we give a proof Lemma 2. We need the following result.

Proposition 2.1. *Let R be a Cohen-Macaulay commutative ring and let P be a prime ideal in R with $\text{height } P \geq g \geq 1$. Then there exists an R -sequence $x_1, \dots, x_g \in P$ such that $\frac{x_1}{1}, \dots, \frac{x_g}{1}$ is part of minimal generators of PR_P in the local ring R_P .*

Proof. We prove the result by induction on g . We first consider the case when $g = 1$. Consider $P^{(2)} = P^2 R_P \cap R$. Note that $P^{(2)} \subseteq P$. If $P \subseteq P^{(2)}$ then $P = P^{(2)}$. So $PR_P = P^2 R_P$. By Nakayama Lemma $PR_P = 0$. Thus $\text{height } P = \dim R_P = 0$ a contradiction since $\text{height } P \geq g = 1$. Let Q_1, \dots, Q_s be minimal primes of R . Since R is Cohen-Macaulay they are also all the associate primes of R . So $P \not\subseteq Q_i$ for $i = 1, \dots, s$. Also we have shown that $P \not\subseteq P^{(2)}$. So by prime avoidance, [4, Lemma 3.3] there exists

$$x_1 \in P \setminus \left(P^{(2)} \cup \left(\bigcup_{i=1}^s Q_i \right) \right).$$

Clearly x_1 is R -regular. Also $\frac{x_1}{1} \notin P^2 R_P$ since $x_1 \notin P^{(2)}$. Thus $\frac{x_1}{1}$ is part of minimal system of generators of PR_P .

We assume the result for $g = i$ and prove the result for $g = i + 1$. Let P be a prime ideal with height $P \geq g = i + 1$. By induction hypotheses there exists $x_1, \dots, x_i \in P$ such that x_1, \dots, x_i is a R -regular sequence and $\frac{x_1}{1}, \dots, \frac{x_i}{1}$ is part of minimal generators of PR_P . Clearly $\text{height}(x_1, \dots, x_i) = i$. Let

$$\{Q_1, \dots, Q_r\} = \text{Min} \frac{R}{(x_1, \dots, x_i)} = \text{Ass} \frac{R}{(x_1, \dots, x_i)}.$$

The last equality holds since R is Cohen-Macaulay. By assumption $\text{height } P \geq i + 1$. So $P \not\subseteq Q_j$ for all $j = 1, \dots, r$. Set

$$J = \langle \frac{x_1}{1}, \dots, \frac{x_i}{1} \rangle R_P + P^2 R_P$$

and $L = J \cap R$. Note that $L \subseteq P$. We claim that $P \not\subseteq L$. Otherwise $P = L$. So $PR_P = J$. Therefore

$$\frac{PR_P}{P^2 R_P} = \langle \frac{x_1}{1}, \dots, \frac{x_i}{1} \rangle.$$

This implies $\dim R_P \leq i$ a contradiction since $\dim R_P = \text{height } P \geq i + 1$. Thus $P \not\subseteq L$. By prime-avoidance there exists

$$x_{i+1} \in P \setminus (L \cup (\cup_{i=1}^r Q_i)).$$

Clearly x_1, \dots, x_{i+1} is a R -regular sequence and $\frac{x_1}{1}, \dots, \frac{x_{i+1}}{1}$ is part of minimal generators of PR_P . Thus by induction our result is true. \square

We now give

Proof of Lemma 2. Let P be a prime ideal in R . Set $\kappa(P) = R_P/PR_P$. We first show

$$(\dagger) \quad \text{Ext}_{R_P}^1(\kappa(P), E_P) = 0.$$

First consider the case when P is a minimal prime of R . Since R is a regular ring, R_P is a field. So $PR_P = 0$. Thus $\kappa(P) = R_P$. Clearly

$$\text{Ext}_{R_P}^1(R_P, E_P) = 0.$$

Suppose $\text{height } P = g \geq 1$. Then by Proposition 2.1 there exists $x_1, \dots, x_g \in P$ such that x_1, \dots, x_g is a R -regular sequence and $\frac{x_1}{1}, \dots, \frac{x_g}{1}$ is part of minimal generators of PR_P . Set $J = (x_1, \dots, x_g)$. Since R is regular, R_P is a regular local ring. It follows that PR_P is minimally generated by g elements. Thus $JR_P = PR_P$. By our hypothesis $\text{Ext}_R^1(R/J, E) = 0$. Localizing we get

$$\text{Ext}_{R_P}^1(\kappa(P), E_P) = \text{Ext}_{R_P}^1(R_P/JR_P, E_P) = 0.$$

Thus we have shown (\dagger) for every prime ideal P of R .

Let $0 \rightarrow E \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ be a minimal injective resolution of E . We have, see [9, 18.7],

$$I^1 = \bigoplus_{P \in \text{Spec}(R)} E(R/P)^{\mu_1(P, M)} \quad \text{where } \mu_1(P, M) = \dim_{\kappa(P)} \text{Ext}_{R_P}^1(\kappa(P), E_P).$$

So $I^1 = 0$. It follows that $E \cong I^0$. Thus E is an injective R -module. \square

3. PROOF OF THEOREM 1

In this section we give a proof of Theorem 1. We also give application of our result to our problem local cohomology. We need the following lemma

Lemma 3.1. *Let A be a commutative ring and let P be a projective A -module. Let x_1, \dots, x_n be a A -regular sequence. Then x_1, \dots, x_n is a P -regular sequence.*

Proof. Let Q be a A -module with $P \oplus Q = F$, free. Clearly x_1, \dots, x_n is a F -regular sequence. It follows that x_1, \dots, x_n is a weak P -regular sequence. It remains to prove that $(\mathbf{x})P \neq P$. Since $A \neq (\mathbf{x})$ it follows that there exists a maximal ideal \mathfrak{m} of A containing (\mathbf{x}) . Since $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module, see [9, Theorem 2.5], we have that $(\mathbf{x})P_{\mathfrak{m}} \neq P_{\mathfrak{m}}$. So $(\mathbf{x})P \neq P$. \square

Remark 3.2. Let A be a commutative ring. Suppose M is a right A -module and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence in A . Then for the Koszul complex $\mathbf{K}_{\bullet}(\mathbf{x}; M)$ we consider elements in $K(\mathbf{x}; M)_i$ as "row's" and not as columns as is the practice when M is a left R -module. This is natural since M is a right A -module. Also note that the maps in $K(\mathbf{x}; M)$ are transposes of the usual maps when M is a left R -module.

Proof of Theorem 1. By Lemma 2 it suffices to show that $\text{Ext}_R^1(R/J, E) = 0$ for every ideal J generated by regular sequence.

Let $J = (a_1, \dots, a_g)$ where a_1, \dots, a_g is a R -regular sequence and let $\phi: J \rightarrow E$ be a R -linear map. We want to prove that there exists R -linear map $\tilde{\phi}: R \rightarrow E$ with $\tilde{\phi}_J = \phi$.

Set $L = SJ$ the left ideal in S generated by J . Define

$$\begin{aligned} \psi: L &\rightarrow E \\ x \in L; \text{ if } x &= s_1 a_1 + \dots + s_g a_g \text{ then} \\ \psi(x) &= s_1 \phi(a_1) + \dots + s_g \phi(a_g). \end{aligned}$$

Sub-Lemma: ψ is well-defined, i.e.,

$$\text{if } x = s_1 a_1 + \dots + s_g a_g = t_1 a_1 + \dots + t_g a_g$$

$$\text{then } s_1 \phi(a_1) + \dots + s_g \phi(a_g) = t_1 \phi(a_1) + \dots + t_g \phi(a_g).$$

We first assume Sub-Lemma. Note that ψ is S -linear. For if $x = s_1 a_1 + \dots + s_g a_g$ then for $t \in S$ we have $tx = ts_1 a_1 + \dots + ts_g a_g$. So

$$\psi(tx) = ts_1 \phi(a_1) + \dots + ts_g \phi(a_g) = t\psi(x).$$

Since E is an injective S -module, there exists an S -linear map $\tilde{\psi}: S \rightarrow E$ with $\tilde{\psi}_L = \psi$.

Note that $\psi_J = \phi$; for if $a \in J$ then $a = r_1 a_1 + \dots + r_g a_g$ for some $r_1, \dots, r_g \in R$. So

$$\psi(a) = r_1 \phi(a_1) + \dots + r_g \phi(a_g) = \phi(a), \quad \text{since } \phi \text{ is } R\text{-linear.}$$

Let $i: R \rightarrow S$ be the inclusion map. Set $\tilde{\phi} = \tilde{\psi} \circ i$. Note that $\tilde{\phi}: R \rightarrow E$ is R -linear and clearly $\tilde{\phi}_J = \phi$. Thus it remains to prove the sub-lemma.

Proof of Sub-Lemma: By Lemma 3.1 we have that a_1, \dots, a_g is a S -regular sequence; here S is considered as a right R -module. For the convenience of the reader we first give the proof when $g = 1, 2$ and then give a general argument.

First consider the case when $g = 1$. So $J = (a_1)$ and $L = SJ = Sa_1$. Let $x \in L$. If $x = s_1a_1 = t_1a_1$, then as a_1 is S -regular we have $s_1 = t_1$. So $s_1\phi(a_1) = t_1\phi(a_1)$.

Next consider the case when $g = 2$. So $J = (a_1, a_2)$ and $L = SJ$. Let $x \in L$. If

$$x = s_1a_1 + s_2a_2 = t_1a_1 + t_2a_2;$$

then as a_1, a_2 is a S -regular sequence there exists $c \in S$ with $s_1 = t_1 - ca_2$ and $s_2 = t_2 + ca_2$. Thus we have

$$s_1\phi(a_1) + s_2\phi(a_2) = t_1\phi(a_1) + t_2\phi(a_2) - c(a_2\phi(a_1) - a_1\phi(a_2)).$$

Since ϕ is R -linear we have that $a_2\phi(a_1) - a_1\phi(a_2) = 0$. Thus the result follows when $g = 2$.

For the general argument, consider the Koszul complex $\mathbf{K}_\bullet(\mathbf{a}; S)$. Before proceeding further please read Remark 3.2. Let $J = (a_1, \dots, a_g)$, with $g \geq 2$, and let $L = SJ$. Let the Koszul maps be given as

$$\dots S^{(g)} \xrightarrow{\psi_g} S^g \xrightarrow{\phi_g} S \rightarrow 0$$

Let $x \in L$. If

$$x = s_1a_1 + s_2a_2 + \dots s_ga_g = t_1a_1 + t_2a_2 + \dots t_ga_g;$$

then $u = [s_1 - t_1, s_2 - t_2, \dots, s_g - t_g] \in \ker \phi_g$. Since \mathbf{a} is a S -regular sequence; $\mathbf{K}_\bullet(\mathbf{a}; S)$ is acyclic. So $u \in \text{image } \psi_g$. Say $u = c\psi_g$ for some $c \in S^{(g)}$. Let

$$w_g = [\phi(a_1), \dots, \phi(a_g)]^{tr} \quad \text{where "tr" denotes transpose.}$$

Then note that

$$\sum_{i=1}^g (s_i - t_i)\phi(a_i) = uw_g = c\psi_g w_g.$$

Thus it is sufficient to prove $\psi_g w_g = 0$. This we prove by induction on g where $g \geq 2$. We first consider the case when $g = 2$. Notice that

$$\psi_2 = [-a_2, a_1].$$

So $\psi_2 w_2 = -a_2\phi(a_1) + a_1\phi(a_2) = 0$, since ϕ is R -linear.

We assume the result when $g = r - 1$ and prove it when $g = r$; (here $r \geq 3$). Notice that

$$\psi_r = \begin{pmatrix} -a_2 & a_1 & 0 & 0 \dots 0 & 0 \\ -a_3 & 0 & a_1 & 0 \dots 0 & 0 \\ \dots & & & & \\ -a_r & 0 & 0 & 0 \dots 0 & a_1 \\ 0 & \widetilde{\psi_{r-1}} & & & \end{pmatrix}$$

Here $\widetilde{\psi_{r-1}}$ is the map in degree 2 of the Koszul complex on S with respect to a_2, \dots, a_r . Set $\widetilde{w_{r-1}} = [\phi(a_2), \dots, \phi(a_r)]^{tr}$. By induction hypothesis we have that $\widetilde{\psi_{r-1}} \widetilde{w_{r-1}} = 0$. Notice

$$\psi_r w_r = \begin{pmatrix} -a_2\phi(a_1) + a_1\phi(a_2) \\ -a_3\phi(a_1) + a_1\phi(a_3) \\ \dots \\ -a_r\phi(a_1) + a_1\phi(a_r) \\ \widetilde{\psi_{r-1}} \widetilde{w_{r-1}} \end{pmatrix}$$

Since ϕ is R -linear and $\widetilde{\psi_{r-1}} \widetilde{w_{r-1}} = 0$ we get that $\psi_r w_r = 0$. □

As an application of our result we prove the following:

Proposition 3.3. *Let K be a field of characteristic zero, $R = K[X_1, \dots, X_n]$ and let I, J be ideals in R . Let $A_n(K) = K \langle X_1, \dots, X_n, \partial_1, \dots, \partial_n \rangle$ be the n^{th} Weyl algebra over K . Then the Mayer-Vietoris sequence*

$$\cdots \rightarrow H_{I+J}^i(R) \xrightarrow{\rho^i} H_I^i(R) \oplus H_J^i(R) \xrightarrow{\pi^i} H_{I \cap J}^i(R) \xrightarrow{\delta^i} H_{I+J}^{i+1}(R) \rightarrow \cdots$$

is a sequence of $A_n(K)$ -modules.

Proof. Set $S = A_n(K)$. Recall that for an R -module N and an ideal I in R the I -torsion module of N is

$$\Gamma_I(N) = \{m \in N \mid I^s m = 0 \text{ for some } s \geq 1; s \text{ depending on } m\}.$$

Let E^\bullet be an injective resolution of R considered as a S -module. Note that the following sequence of complexes of R -modules

$$0 \rightarrow \Gamma_{I+J}(E^\bullet) \rightarrow \Gamma_I(E^\bullet) \oplus \Gamma_J(E^\bullet) \rightarrow \Gamma_{I \cap J}(E^\bullet) \rightarrow 0,$$

is exact, see [5, page 154]. It can be easily verified that if M is a S -module then $\Gamma_I(M)$ is a S -submodule of M . Thus the above sequence is an exact sequence of a complex of S -modules. The result follows. \square

4. INDECOMPOSABLE INJECTIVE S -MODULES

4.1. Assumptions: In this section we assume that S is a left Noetherian ring containing a regular commutative ring R as a subring. Assume that S considered as a right R -module is projective. We make a further assumption:

(*) Given an ideal I in R and $s \in S$, there exists $r \geq 1$ (r depending on s) such that $I^r s \subseteq SI$.

In Theorem 1 we proved that every injective left S -module E is an injective R -module. In this section we investigate the structure of E .

Remark 4.2. The assumption (*) above is trivially satisfied if $R \subseteq Z(S)$. So our examples 4,5, 8, 10, 11, 12 trivially satisfy our hypothesis. It takes a little work to show that examples 1,2,3, 7, 9 satisfy (*). We prove it for example 7. The other cases being similar. Let R be any commutative regular ring and let $\delta: R \rightarrow R$ be a derivation on R . Consider the differential polynomial ring $S = R[X; \delta]$. An element s in S has a unique expression $s = \sum_{i=0}^m a_i X^i$. It suffices to take $s = aX^m$. Note that $Xb = bX + \delta(b)$ for any $b \in R$. We can prove that

$$bX^n = \sum_{i=0}^n (-1)^i \binom{n}{i} X^{n-i} \delta^i(b).$$

So if $b \in I^{m+1}$ then $\delta^i(b) \in I$ for all $i = 0, \dots, m$. Notice that

$$bs = baX^m = abX^m = \sum_{i=0}^m (-1)^i \binom{m}{i} aX^{m-i} \delta^i(b) \in SI.$$

Thus $I^{m+1}s \subseteq SI$.

The significance of our assumption (*) is the following result.

Proposition 4.3. *(with assumptions as in 4.1). For any ideal I in R and a S -module M we have that $\Gamma_I(M)$ is a S -submodule of M .*

Proof. Clearly $\Gamma_I(M)$ is a R -submodule of M . So let $s \in S$ and $m \in \Gamma_I(M)$. Say $I^l m = 0$. Set $J = I^l$. By our assumption (*) we have that there exists $r \geq 1$ such that $J^r s \subseteq SJ$. So $J^r sm = 0$. Thus $I^{r+l} sm = 0$. Therefore $sm \in \Gamma_I(M)$. \square

4.4. Disussion: Let M be a S -module and consider the injective hull $E_S(M)$. Now $\text{Ass}_R E_S(M)$ is a non-empty set and as R is Noetherian $\text{Ass}_R E_S(M)$ has maximal elements with respect to inclusion.

Proposition 4.5. *(with assumptions as in 4.1). Let $M \subseteq N$ be an essential extension of S -modules. Let $P \in \text{Ass}_R N$ be a maximal element in $\text{Ass}_R N$. Then $P \in \text{Ass}_R M$. In particular this result holds when $N = E_S(M)$, the injective hull of M as a S -module.*

Proof. Let $P = (0: x)$ for some non-zero $x \in N$. Notice $Sx \supseteq Rx \neq 0$. Since N is an essential extension of M we have that $Sx \cap M \neq 0$. So there exists $s \in S$ with $sx \in M$ and $sx \neq 0$. Let

$$\mathcal{F} = \{(0: t) \mid t \in N, t \neq 0\}$$

Maximal elements in \mathcal{F} are the associate primes of N . Note that $(0: sx) \in \mathcal{F}$. So $(0: sx) \subseteq Q$ for some $Q \in \text{Ass}_R N$.

By our assumption (*) we have that $P^r s \subseteq SP$ for some $r \geq 1$. As $Px = 0$ we obtain that $P^r sx = 0$. Thus $P^r \subseteq Q$. Since Q is prime we have that $P \subseteq Q$. By our choice of P we have that $Q = P$.

Set $y = sx$. Thus we have $P^r \subseteq (0: y) \subseteq P$. We localize at P . So we have

$$P^r R_P \subseteq (0: {}_{R_P} y) \subseteq P R_P.$$

As $P^r R_P y = 0$ and $y \neq 0$, there exists $i \geq 1$ with $P^{i-1} R_P y \neq 0$ and $P^i R_P y = 0$. Let $u \in P^{i-1} R_P y$ be non-zero. Then $P R_P u = 0$. Since $P R_P$ is maximal ideal in R_P we get that $(0: u) = P R_P$. Thus $P R_P$ is associate to M_P . Therefore P is an associate prime of M . \square

Since S is left Noetherian, every injective left module over S is a direct sum of indecomposable injective modules. We prove Theorem 3. For the convenience of the reader we restate it here.

Theorem 4.6. *(with assumptions as in 4.1). Let M be an S -module with $E_S(M)$ an indecomposable injective S -module. Then $\text{Ass}_R E_S(M)$ has a unique maximal element P . Furthermore $P \in \text{Ass}_R M$.*

Proof. Let P, Q be maximal associate primes of $E_S(M)$. We prove that $P = Q$. By 4.5 P is also an associate prime of M . By 4.3 we have that $\Gamma_P(M)$ is a S -submodule of M . Since P is an associate prime of M we have that $\Gamma_P(M) \neq 0$. Furthermore as P is a maximal associate prime of M it can be easily verified that $\text{Ass}_R \Gamma_P(M) = \{P\}$. By [8, 2.2], we have that $E_S(M) = E_S(\Gamma_P(M))$. Since Q is a maximal associate prime of $E_S(M)$, by 4.5 we get $Q \in \text{Ass}_R \Gamma_P(M) = \{P\}$. So $Q = P$. \square

Next we consider the injective hull of simple S modules. Recall that an S -module M is simple if the only submodules of M are 0 and M . It is easy to see that M is simple if and only if $M \cong S/J$ where J is a maximal left ideal in S .

Proposition 4.7. *Let J be a maximal left ideal in S . Then $I = J \cap R$ is a primary ideal in R . Say I is P -primary. Then $\text{Ass}_R S/J = \{P\}$. Furthermore P is the unique maximal element of $\text{Ass}_R E_S(S/J)$.*

Proof. Suppose $ab \in I$ and $a \notin I$. So $a \notin J$. Thus $J + Sa = S$. So $b = j + sa$ for some $j \in J$ and $s \in S$. Set $K = Rb$. Then by our assumption (*), there exists $r \geq 1$ such that $b^r s \in SK$. So $b^r s = \sum_{i=1}^l s_i b = db$ for some $s_i, d \in S$. Thus $b^{r+1} = b^r j + dba$. So we get that $b^{r+1} \in J$. Thus $b^{r+1} \in I$. Therefore I is a primary ideal in R .

Say I is P -primary. So $P \in \text{Ass}_R R/I \subseteq \text{Ass}_R S/J$. We claim that $\text{Ass}_R S/J = \{P\}$. Let Q be a maximal element of $\text{Ass}_R S/J$. Then $\text{Ass}_R \Gamma_Q(S/J) = \{Q\}$. Furthermore by 4.3 we have that $\Gamma_Q(S/J)$ is a S -submodule of S/J . As S/J is simple and $\Gamma_Q(S/J) \neq 0$ we have that $\Gamma_Q(S/J) = S/J$. Thus it follows that $Q = P$ and $\text{Ass}_R S/J = \{P\}$. By 4.6, P is the unique maximal element of $\text{Ass}_R E_S(S/J)$. \square

When R is in the center of S then we can say more.

Theorem 4.8. *(with assumptions as in 4.1) Further assume that $R \subseteq Z(S)$. Let M be a S -module. Then $\text{Ass}_R M = \text{Ass}_R E_S(M)$.*

The main ingredient of 4.8 is the following:

Lemma 4.9. *Let $R \subset S$ be an extension of rings with S left Noetherian and R commutative. Assume $R \subseteq Z(S)$. Let M, N be left S -modules with $M \subseteq N$ an essential extension of S -modules. Let T be any multiplicatively closed subset of R . Then $T^{-1}M \subseteq T^{-1}N$ is an essential extension of left $T^{-1}S$ -modules.*

The proof of Lemma 4.9 is similar to [3, 3.2.5]. The assumption $R \subseteq Z(S)$ is used to conclude that $\text{ann}_S(x) \subseteq \text{ann}_S(tx)$ for any $x \in N$ and $t \in T$. We now give

Proof of Theorem 4.8. As $M \subseteq E_S(M)$ we have that $\text{Ass}_R M \subseteq \text{Ass}_R E_S(M)$. Conversely let $P \in \text{Ass}_R E_S(M)$. Set $T = R \setminus P$. Consider the extension $T^{-1}R \subseteq T^{-1}S$. Notice that $T^{-1}R \subseteq Z(T^{-1}S)$. It follows that the extension $T^{-1}R \subseteq T^{-1}S$ satisfies our assumptions 4.1. By Lemma 4.9 we get that $T^{-1}E_S(M)$ is an essential extension of $T^{-1}M$. Clearly $PT^{-1}R$ is a maximal element of $\text{Ass}_{T^{-1}R} T^{-1}E_S(M)$. So by 4.5 we get that $PT^{-1}R \in \text{Ass}_{T^{-1}R} M$. Thus $P \in \text{Ass}_R M$. \square

5. THE CASE WHEN S IS COMMUTATIVE AND A FINITE EXTENSION OF R

In this section R, S are commutative and S a finite extension of R and a projective R -module. Recall that we assume R to be regular. It is easily seen that S is a Cohen-Macaulay ring. In-fact S is a projective R -module if and only if S is Cohen-Macaulay.

We now give a proof of Theorem 4. For convenience of the reader we restate it here.

Theorem 5.1. *(with assumptions as above). Let \mathfrak{m} be a maximal ideal of S . Set $\mathfrak{n} = \mathfrak{m} \cap R$. Let $\mathfrak{n}S = Q_1 \cap Q_2 \cap \cdots \cap Q_c$ be a minimal primary decomposition of $\mathfrak{n}S$ where Q_1 is \mathfrak{m} -primary. Then*

$$E_S \left(\frac{S}{\mathfrak{m}} \right) = E_R \left(\frac{R}{\mathfrak{n}} \right)^{\ell_R(S/Q_1)}.$$

Proof. Notice that \mathfrak{n} is a maximal ideal of R . Set $E = E_S(S/\mathfrak{m})$. By Theorem 1 we have that E is an injective R -module.

Note that we have an injection $R/\mathfrak{n} \hookrightarrow S/\mathfrak{m}$ of R -modules and an injection $S/\mathfrak{m} \hookrightarrow E$ of S -modules. Composing we have an injection $R/\mathfrak{n} \hookrightarrow E$ of R -modules. Thus $\mathfrak{n} \in \text{Ass}_R E$.

Claim 1: $\text{Ass}_R E = \{\mathfrak{n}\}$.

Suppose $P \in \text{Ass}_R E$. Say $P = (0 : x)$ for some non-zero $x \in E$. Each element of E is annihilated by a power of \mathfrak{m} . Say $\mathfrak{m}^t x = 0$. Then $\mathfrak{n}^t x = 0$. This implies $\mathfrak{n}^t \subseteq P$. As P is prime we obtain $\mathfrak{n} \subseteq P$. So $P = \mathfrak{n}$. This proves Claim 1.

Thus by the structure theorem of injectives [3, 3.2.8] we have that

$$E = E_R \left(\frac{R}{\mathfrak{n}} \right)^c \quad \text{where } c = \dim_{R/\mathfrak{n}} \text{Hom}_R \left(\frac{R}{\mathfrak{n}}, E \right).$$

Notice that

$$\text{soc}_{R,\mathfrak{n}} E = \{t \in E \mid \mathfrak{n}t = 0\} \cong \text{Hom}_R \left(\frac{R}{\mathfrak{n}}, E \right).$$

Claim 2: $\text{soc}_{R,\mathfrak{n}} E = (0 :_E Q_1)$.

Let $t \in \text{soc}_{R,\mathfrak{n}} E$. Then note that $\mathfrak{n}St = 0$. Notice $Q_2 \cap \cdots \cap Q_c \not\subseteq \mathfrak{m}$. Say $\xi \in Q_2 \cap \cdots \cap Q_c \setminus \mathfrak{m}$. Let $a \in Q_1$. So $a\xi \in \mathfrak{n}S$. Thus $a\xi t = 0$. Now $\text{Ass}_S E = \{\mathfrak{m}\}$. This implies that ξ is a non-zero divisor on E . So $at = 0$. Thus $Q_1 t = 0$.

Conversely if $t \in (0 :_E Q_1)$, then $Q_1 t = 0$. Since $\mathfrak{n} \subseteq \mathfrak{n}S \subseteq Q_1$ we get that $\mathfrak{n}t = 0$. Thus Claim 2 is proved.

Notice $(0 :_E Q_1) \cong \text{Hom}_S(S/Q_1, E)$. Since $\text{Supp}_S(S/Q_1) = \{\mathfrak{m}\}$ and as $E = E_S(S/\mathfrak{m})$ we get that $\ell_S(\text{Hom}_S(S/Q_1, E)) = \ell_S(S/Q_1)$.

Notice that since $\mathfrak{n}S \subseteq Q_1 \subseteq \mathfrak{m}$ we have that $Q_1 \cap R = \mathfrak{n}$. So S/Q_1 is a finite dimensional vector space over R/\mathfrak{n} . Again as $\text{Supp}_S(S/Q_1) = \{\mathfrak{m}\}$ we get that $\ell_R(S/Q) = r\ell_S(S/Q_1)$ where $r = \dim_{R/\mathfrak{n}} S/\mathfrak{m}$.

Thus

$$c = \dim_{R/\mathfrak{n}} \text{soc}_{R,\mathfrak{n}} E = \dim_{R/\mathfrak{n}} (0 :_E Q_1) = r\ell_S(S/Q_1) = \ell_R(S/Q_1).$$

□

REFERENCES

- [1] S. M. Bhatwadekar, Cancellation theorems for projective modules over a two-dimensional ring and its polynomial extensions. *Compositio Math.* 128 (2001), no. 3, 339-359.
- [2] J.-E. Björk, *Rings of differential operators*. North-Holland Mathematical Library, 21. North-Holland Publishing Co., Amsterdam-New York, 1979.
- [3] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, revised edition, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, 1998.
- [4] D. Eisenbud, *Commutative algebra; With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- [5] S. B. Iyengar; G. J. Leuschke; A. Leykin; C. Miller; E. Miller; A. K. Singh and U. Walther, *Twenty-four hours of local cohomology*. Graduate Studies in Mathematics, 87. American Mathematical Society, Providence, RI, 2007.
- [6] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). *Invent. Math.* 113 (1993), no. 1, 41-55.
- [7] H. B. Mann, On integral bases. *Proc. Amer. Math. Soc.* 9 (1958), 167-172.
- [8] E. Matlis, Injective modules over Noetherian rings. *Pacific J. Math.* 8 (1958), 511-528.
- [9] H. Matsumura, *Commutative ring theory*. Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.
- [10] R. MacKenzie and J. Scheuneman, A number field without a relative integral basis. *Amer. Math. Monthly.* 78 (1971), 882-883.
- [11] T. J. Puthenpurakal, Koszul homology of local cohomology modules, *Preprint*.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI
400 076

E-mail address: `tputhen@math.iitb.ac.in`